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CHARACTERISTIC ROOTS AND THE CORRESPONDING VECTORS

Question No. 1:

Find the characteristic roots and the corresponding characteristic vectors of the matrix. $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Aim :

To find the characteristic roots and the corresponding characteristic vectors of the given matrix.

Formula :

1. If A is a square matrix of order n and I is the n x n unit matrix then $|A - \lambda I| = 0$. Where λ is a scalar, is called the characteristic equation of A and the roots of the characteristic equation are the characteristic roots of A.
2. Let A be a square matrix, then a non-zero vector X is called a characteristic vector of A, if there exists a scalar λ such that $AX = \lambda X$.

Procedure :

Given that $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$$

$$\Rightarrow (8-\lambda)[\lambda^2 - 10\lambda + 5] + 6[6\lambda - 10] + 2[2\lambda + 10]$$

$$\Rightarrow 8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 36\lambda - 60 + 4\lambda + 20 = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow -\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\Rightarrow \lambda = 0, 3, 15$$

Hence the characteristic roots of A are 0,3,15.

1. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a characteristic vector corresponding to λ =0.

Then AX = λX ⇒ AX = 0X = 0

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \xrightarrow{R_{1,3}} \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underset{R_3 - 4R_1}{\xrightarrow{R_2 + 3R_1}} \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \underset{R_3 \times 1/10}{\xrightarrow{R_2 \times 1/5}} \begin{bmatrix} 2 & -4 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & -4 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The above equations are $2x_1 - 4x_2 + 3x_3 = 0$ and $-x_2 + x_3 = 0$.

If $x_2 = K$ then $x_3 = K$

$\therefore 2x_1 = 4K - 3K = K$

$\therefore x_1 = \frac{K}{2}$

Hence the characteristic vectors corresponding to λ = 0 are $K\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ where K is a non-zero scalar.

2. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a characteristic vector corresponding to λ =3. Then AX =3X

⇒ (A-3I)X=0

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_{1,3}} \begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{matrix} R_1 \times 1/2 \\ \longrightarrow \\ R_2 \times 1/2 \end{matrix} \begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -2 \\ 5 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{matrix} R_2 + 3R_1 \\ \longrightarrow \\ R_3 - 5R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_1 \times 3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_1 \times \frac{-1}{2}} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence the equations are $x_1 - 2x_2 = 0$

$$2x_2 + x_3 = 0$$

If $x_2 = K$, then $x_3 = -2K$ and $x_1 = 2K$

Hence the characteristic vectors corresponding to λ =3 are $K \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ where K is a non-zero scalar.

3. Let λ=15 and X= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a characteristic vector corresponding to λ = 15.

∴ AX = 15 X

⇒ (A – 15 I) X = 0

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_{1,3}}$$

$$\begin{bmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{matrix} R_1 \times \frac{1}{2} \\ \longrightarrow \\ R_2 \times \frac{1}{2} \end{matrix} \begin{bmatrix} 1 & -2 & -6 \\ -3 & -4 & -2 \\ -7 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{matrix} R_2 + 3.R_1 \\ \longrightarrow \\ R_3 + 7.R_1 \end{matrix} \begin{bmatrix} 1 & -2 & -6 \\ 0 & -10 & -20 \\ 0 & -20 & -40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{matrix} R_2 \times -\frac{1}{10} \\ \longrightarrow \\ R_3 \times -\frac{1}{20} \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence the equations are $x_1 - 2x_2 - 6x_3 = 0$

$$x_2 + 2x_3 = 0$$

If $x_3 = K$ then $x_2 = -2K$

$x_1 = -4K + 6K = 2K$

Hence the characteristic vectors corresponding to λ=15 are $K \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ where K is a non-zero scalar.

Conclusion : The characteristic vectors of A are 0, 3,15.

1. The characteristic vectors corresponding to λ = 0 are $K \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ where K is a non-zero scalar.
2. The characteristic vectors corresponding to λ = 3 are $K \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ where K is a non-zero scalar.
3. The characteristic vectors corresponding to λ = 15 are $K \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ where K is a non-zero scalar.

Question No. 2:

Find the characteristic roots and the corresponding characteristic vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Aim : To find the characteristic roots and the corresponding characteristic vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Formula :

1. If A is a square matrix of order n and I is the n x n unit matrix then |A – λI | = 0. Where λ is a scalar, is called the characteristic equation of A and the roots of the characteristic equation are the characteristic roots of A.
2. Let A be a square matrix, then a non-zero vector X is called a characteristic vector of A, if there exists a scalar λ such that AX = λX.

Procedure :

Given that $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is |A – λ I|=0

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$$

$$\Rightarrow 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\Rightarrow -\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\Rightarrow (\lambda - 2)^2(\lambda - 8) = 0$$

$$\Rightarrow \lambda = 2, 2, 8$$

1. Let $\lambda = 2$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a characteristic vector corresponding to $\lambda = 2$

∴ AX = 2X ⇒ (A – 2I) X = 0

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_3 + R_2}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_1 \times 1/2} \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence the equations are $2x_1 - x_2 + x_3 = 0$

Let $x_1 = K_1$and $x_2 = x_2$ then $x_3 = -2K_1 + K_2$

Where $K_1$and x_2 are non-zero scalars.

Hence the characteristic vectors corresponding

to λ = 2 are $\begin{bmatrix} K_1 \\ K_2 \\ -2K_1 + K_2 \end{bmatrix} i.e\ K_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

2. Let λ = 8. Let X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a characteristic vector corresponding to λ = 8

Then AX = 8X ⇒ (A – 8 I) X = 0

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{matrix} R_2 - R_1 \\ \longrightarrow \\ R_3 + R_1 \end{matrix} \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_3 + R_2}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \begin{matrix} R_1 \times -\frac{1}{2} \\ \longrightarrow \\ R_3 \times -\frac{1}{3} \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence the equations are $x_1 + x_2 - x_3 = 0; x_2 + x_3 = 0$

If $x_3 = K$ then $K_2 = -K$

$\therefore x_1 = K + K = 2K$

Hence the characteristic vectors corresponding to λ = 8 are $K\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ where K is non-zero scalar.

Conclusion : The characteristic roots of the given matrix are 2,2,8.

1. The characteristic vectors corresponding to λ = 2 are $K_1\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + K_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ where K_1 and K_2 are non-zero scalars.
2. The characteristic vectors corresponding to λ = 8 are $K\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ where K is non-zero scalar.

Question No. 3:

Find the characteristic roots and the corresponding vectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Aim : To find the characteristic roots and the corresponding the characteristic vectors of the given matrix.

Formula :

1. If A is a square matrix of order n and I is the n x n unit matrix then |A – λI | = 0. Where λ is a scalar, is called the characteristic equation of A and the roots of the characteristic equation are the characteristic roots of A.

2. Let A be a square matrix, then a non-zero vector X is called a characteristic vector of A, if there exists a scalar λ such that AX = λX.

Procedure : Given that $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

The characteristic equation of A is | A - λI| =0

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda)^3 = 0.$

$\Rightarrow \lambda = 2, 2, 2.$

1. Let $\lambda = 2$ Let X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a characteristic vector corresponding to λ =2.

Then AX = 2X $\Rightarrow$ (A – 2 I) x = 0

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence $x_2 = 0$; $x_3 = 0$

We take $x_1 = K$; Where K is a non-zero scalar. Hence the characteristic vectors corresponding to $\lambda = 2$ are $K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where $0 \neq K \in \mathbb{R}$.

Conclusion:

The characteristic roots of A are 2,2,2.

The characteristic vectors corresponding to 2 are $K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where K is a non-zero scalar.

Exercise

Find the characteristic roots and corresponding characteristic vectors of the following matrices.

1. $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 6 & 8 \\ 0 & 7 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

CAYLEY – HAMILTON THEOREM

Question No. 1:

If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$, verify Cayley-Hamilton theorem and hence find A^{-1}

Formula : To show that A satisfy its characteristic equation and hence we find A^{-1}

Formula : Every square matrix satisfies its characteristic equation.

Procedure :

Given that $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

Then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(-2-\lambda)] - 1[+5(-2-\lambda)+3] + 2[3-\lambda] = 0$$

$$\Rightarrow (2-\lambda)[\lambda^2 - \lambda - 6] - [-5\lambda - 7] + 6 - 2\lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$$

We shall prove that A satisfy its characteristic equation.

i.e. $A^3 - 3A^2 - 7A - I = 0$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

$Now\ A^3 - 3A^2 - 7A - I$

$$\begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} + \begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix} + \begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 70 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A satisfy its characteristic equation.

Now $A^3 - 3A^2 - 7A - I = 0$

$\Rightarrow A^{-1}(A^3 - 3A^2 - 7A - I) = A^{-1}.0$

$\Rightarrow A^2 - 3A - 7I - A^{-1} = 0$

$\therefore A^{-1} = A^2 - 3A - 7I$

$$= \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ -15 & -9 & -9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

Conclusion : A satisfy characteristic equation and $A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$

Question No. 2 : Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$, verify Cayley –Hamilton theorem and hence find A^{-1}

Aim :

To verify Cayley –Hamilton theorem and hence to find A^{-1}

Formula :

Every square matrix is a root of its characteristic polynomial.

Procedure :

Given that $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$

Characteristic polynomial of A is $|A - \lambda I|$

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)[(5-\lambda)^2 - 1] - 1[-5+\lambda+1] + 1[1 - 1(5-\lambda)]$$

$$= (3-\lambda)[\lambda^2 - 10\lambda + 24] + 4 - \lambda + \lambda - 4$$

$$= 3\lambda^2 - 30\lambda + 72 - \lambda^3 + 10\lambda^2 - 24\lambda$$

$$= -\lambda^3 + 13\lambda^2 - 54\lambda + 72$$

Now $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix}$ and

$$A^3 = \begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix}$$

$-A^3 + 13A^2 - 54A + 72\,\mathrm{I}$

$$= -\begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix} + 13\begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} - 54\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} + 72\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -27 & -37 & -37 \\ 63 & -127 & 89 \\ -63 & 63 & -153 \end{bmatrix} + \begin{bmatrix} 117 & 91 & 91 \\ -117 & 335 & -143 \\ 117 & -117 & 351 \end{bmatrix} + \begin{bmatrix} 162 & -54 & -54 \\ 54 & -270 & 54 \\ -54 & 54 & 270 \end{bmatrix}$$

$$+ \begin{bmatrix} 72 & 0 & 0 \\ 0 & 72 & 0 \\ 0 & 0 & 72 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence is A is a root of its characteristic polynomial.

Now $-A^3 + 13A^2 - 54A + 72\,\mathrm{I} = 0$

Multiplying both sides by A^{-1}, we get

$$\Rightarrow 72\,A^{-1} = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} - 13\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} + 54\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{72}\begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

Conclusion : A satisfies its characteristic polynomial and

$$A^{-1} = \frac{1}{72}\begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

Question No. 3 : Using Cayley-Hamilton theorem find A^{-1} and A^4 for the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ +6 & 2 & -1 \end{bmatrix}$

Aim : To find A^{-1} and A^4 for the given matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ +6 & 2 & -1 \end{bmatrix}$

Formula : Every square matrix is a root of its characteristic equation.

Procedure :

Given that $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ +6 & 2 & -1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ +6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$= (7-\lambda)[(-1-\lambda)(-1-\lambda) - 4] - 2[-6(-1-\lambda) - 12]$$

$$-2[-12 - 6\,(-1-\lambda)] = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Now $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ -6 & 2 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix};\ A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

By the Caley –Hamilton Theorem $A^3 - 5A^2 + 7A - 3I = 0.$

Multiplying both sides byA^{-1}, we get

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$\text{Now } A^2 - 5A + 7I = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$$\text{Hence } A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Now $A^3 - 5A^2 + 7A - 3I = 0$

$\Rightarrow A^3 = 5A^2 + 7A + 3I = 0$

$\Rightarrow A^4 = 5A^3 - 7A^2 + 3A$

$$\begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ 168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ 240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Conclusion : For the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ -6 & 2 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix} \text{ and } A^4 = \begin{bmatrix} 241 & 80 & -80 \\ 240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Question No. 4:

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Aim : If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ then to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Formula :

1. If A is a square matrix then its characteristic equation is $|A - \lambda I| = 0$.
2. Every square matrix is a root of its characteristic equation.

Procedure :

Given that $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) + 2 = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

By the Cayley –Hamilton theorem A satisfy its characteristic equation.

$\therefore A^2 - 4A + 5I = 0 \Rightarrow A^2 = 4A - 5\,I$

$\therefore A^3 = 4A^2 - 5A, A^4 = 4A^3 - 5A^2$

$A^5 = 4A^4 - 5A^3, A^6 = 4A^5 - 5A^4$

$\therefore A^6 - 4A^5 + 8A^4 = 3A^4 - 12A^3 - 15A^2$

$\Rightarrow A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = -A^2 = -4A + 5I$

Which is a linear polynomial of A.

Conclusion :

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = -4A + 5I$ which is a linear polynomial in A.

Exercise

Find the characteristic roots and corresponding characteristic vectors of the following matrices.

1. Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley – Hamilton theorem.

2. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

   satisfies characteristic equation. Hence find A^{-1}

3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A

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